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The motion of solids in inviscid uniform vortical fields

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We consider the general motion (translation and rotation) of a deformable or rigid body of arbitrary shape in a linear shear flow of an effectively inviscid and incompressible fluid possessing uniform vorticity. The ambient vorticity may be time-dependent. For two-dimensional configurations a solution with uniform vorticity is possible for all times and for three-dimensional, it is possible only initially or during a short time interval after the body is impulsively introduced into the fluid. General analytic expressions for the vortical force and moment exerted on an arbitrary moving body are presented. Bearing in mind applications for large non-spherical bubble dynamics, the general expressions for the hydrodynamic loads are further reduced for symmetric quadratic shapes such as two-dimensional ellipses or three-dimensional ellipsoids. The simplified expressions are given in terms of the body's added-mass tensor, its six velocities and the ambient vorticity. The few available degenerate solutions for cylinders and spheres are readily obtained as limiting cases.

1. Introduction

A fundamental problem in fluid mechanics, especially in multiphase flows, is to determine the equations of motion and consequently the trajectories of a rigid or deformable body moving in an ambient vortical flow field. It is a common practice in bubble dynamics when the bubble surface is assumed to be clean, to apply a zero-shear boundary condition on its surface when the Reynolds number is relatively high. Thus, by allowing slip and ignoring the effect of boundary-layer vorticity for streamlined shapes, the problem can be treated within the framework of inviscid flow theory. According to Lamb (1932, p. 233), determining the pressure loads exerted on a moving body in an inviscid fluid endowed with vorticity is a problem of considerable interest, but unfortunately not very tractable. The only exception being a two-dimensional case and uniform vorticity. It is often postulated that the ambient vorticity is weak and uniform at infinity. However, even under these assumptions, the only three-dimensional solution available is that for the force acting on a rigid sphere embedded in a simple shear flow (e.g. Auton 1987). Auton's solution, which is based on Lighthill's (1956) idea of using a Lagrangian drift function, is involved and is limited to perfectly symmetric spherical geometries. No solutions have been reported so far for non-spherical shapes, even though some conjectures (as yet unproved) are also made for special axisymmetric bodies. The general impression from reviewing the literature, is that the problem is still far from being solved and that some confusion even arises for the elementary spherical case, with regard to the proper use of the frame-indifference objectivity principle. For this reason, Magnaudet & Eames (2000) have stated in their comprehensive review (see p. 701), that formulating the equations of motion for a general shape moving with high Re in a non-uniform flow field, is the first out of three open problems of particular importance in bubble motion. This proposition serves as one of the prime motivations for the present study.

The problem of evaluating the pressure loads experienced by a body moving in an inviscid flow field possessing uniform vorticity was first posed in the pioneering work of Proudman (1916) following a suggestion of G. I. Taylor. It appears that even to date a general solution to this fundamental problem is not available, in spite of the considerable progress made for planar flows. Proudman noted the difference between two- and three-dimensional cases. Namely, a two-dimensional motion with uniform vorticity is possible for all times, but in three-dimensions such a motion is possible only for a short time after introducing the body into the fluid because of vorticity stretching effects. This point can be further explained by considering the Helmholtz vorticity transport equation for inviscid incompressible flows,

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v}, \tag{1.1}$$

where $\boldsymbol{v}$ is the rotational velocity and $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$ denotes the ambient vorticity. For two-dimensional flows, $\boldsymbol{\omega}\perp\boldsymbol{v}$ and therefore $\mathrm{D}\boldsymbol{\omega}/\mathrm{D}t = 0$, which implies that flows with uniform vorticity are possible. However, in three-dimensional flows the right-hand side of (1.1) is generally non-zero and thus flows with uniform vorticity are possible only initially, or during a short time interval, before vortex stretching effects become dominant.

Two simple cases of effectively inviscid flows with uniform vorticity are discussed in some detail by Batchelor (1967, pp. 538–543). The first considers a body moving in an infinite expanse of fluid which is in rigid rotation at infinity (rotating machinery) and recalling that a rotation with an angular velocity $\frac{1}{2}\boldsymbol{\omega}$, generates a uniform vorticity $\boldsymbol{\omega}$. The second case is that of a body embedded in a fluid in simple shearing motion at infinity, where the ambient velocity varies linearly with the spatial coordinates. It can be shown that the second case of a simple shear, can be obtained as a combination of the first case of rigid-body rotation and an irrotational straining flow. Here, we consider the general case of two-dimensional or three-dimensional deformable shapes moving with six degrees of freedom, i.e. translation velocity $\boldsymbol{U}$ and angular rotation $\boldsymbol{\Omega}$. Analytic expressions are derived for the force and moment acting on the body in terms of its six velocities, the ambient stream $\boldsymbol{V}$ and vorticity $\boldsymbol{\omega}_0$, as well as the body's added-mass tensor.

So far, the only two configurations which have been considered in the literature, are those which preserve perfect symmetry, i.e. a cylinder and a sphere. For these shapes, Auton, Hunt & Prud'homme (1988, equation (2.13)) proposed the following expression for the force exerted on the body,

$$\boldsymbol{F} = B\left\{(1+C_M)\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} + C_L\boldsymbol{\omega}_0\times(\boldsymbol{U}-\boldsymbol{V}) - C_M\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t}\right\}, \tag{1.2}$$

where the fluid density is taken as unity, B is the body volume and C_M is its added-mass (see also Magnaudet & Eames 2000, equation (11)). The lift coefficient C_L is decomposed as follows:

$$C_L(t) = \tfrac{1}{2}(1+C_M) + C_{L\omega}(t), \tag{1.3}$$

where $C_{L\omega}$ is the so-called 'rotational' lift coefficient. This coefficient is generally time-dependent. Legendre & Magnaudet (1998) have shown that the 'initial' value for a sphere is $C_{L\omega}(0) = 0$ and Auton (1987) has demonstrated that $C_{L\omega}(\infty) = -\frac{1}{4}$, under the assumption of weak vorticity. The variation of C_L with t between these two extreme values is depicted in figure 20 of Legendre & Magnaudet (1998). For cylinders, $C_{L\omega} = 1$ and thus $C_L = 2$ for all times, as shown for example in Batchelor (1967, p. 542). It is also shown there that in the case of rigid fluid rotation about a circular cylinder, $C_L = 1$.

It should be emphasized that (1.2) is valid only for cylinders/spheres, which are perfectly symmetric shapes and where the angular velocity $\boldsymbol{\Omega}$ does not play a role as long as the fluid is inviscid. An extension for non-spherical shapes is far from being straightforward even for non-rotating shapes, i.e. $\boldsymbol{\Omega} = 0$. For non-spherical streamlined shapes, the torque acting on the body should be considered also, since together with the force, it determines the equations of motion of the body from which its spatial trajectory can be found. The only previous attempts to generalize the analysis for arbitrary (i.e. non perfectly-symmetric) configurations is Batchelor's (1967, equation (7.4.16)) for two-dimensional cylinders and Miloh's (1994) three-dimensional generalization. It will be demonstrated in the following that these solutions for the vortical force are incomplete. In addition, they do not consider the vortical moment which is an important input parameter in bubble dynamics (e.g. Feng & Leal 1997; Magnaudet & Eames 2000), especially for non-spherical (spheroidal) large bubbles. These issues are further addressed in this work.

The structure of the paper is as follows. In § 2, we consider the general case of a deformable shape moving with six degrees of freedom in an arbitrary linear shear flow with a uniform ambient vorticity $\boldsymbol{\omega}_0(t)$. The induced velocity field is decomposed into 'rotational' and 'irrotational' components and the corresponding Euler equations are derived subject to the proper boundary conditions imposed on the body surface S. The total force and moment acting on the moving body are then split into 'potential' and 'vortical' parts, where the latter is again decomposed into linear and quadratic terms in the ambient vorticity $\boldsymbol{\omega}_0$. Analytic expressions for the 'vortical' force and moment are presented in § 3 for arbitrary two-dimensional and three-dimensional shapes. It is then demonstrated in § 4 that these hydrodynamic loads can be greatly simplified for quadratic shapes, which preserve the sign symmetry $S(X) = S(-X)$, such as two-dimensional (cylinders, ellipses, etc.) and three-dimensional (spheres, spheroids, ellipsoids, etc.). First, we provide the corresponding analytic expressions for both the force $\boldsymbol{F}_\omega$ and moment $\boldsymbol{M}_\omega$, in the case where the flow at infinity is in rigid-body rotation. Then, we consider the case of a simple shear flow with uniform vorticity and derive the appropriate expressions for the 'shear' force and torque. The new relationships are written in terms of the body's added-mass tensor, its six velocities and the ambient uniform vorticity. They provide a unified methodology for treating both two-dimensional and three-dimensional shapes, as well as cases involving rigid rotation and simple shearing motion. The two existing solutions for cylinders and spheres, are readily obtained from the general solution as limiting trivial cases.

2. General formulation

We consider an inviscid and incompressible three-dimensional flow field with a uniform time-dependent vorticity $\boldsymbol{\omega}_0(t)$ which is governed by the Euler equation. The ambient velocity field is taken as a linear shear $\boldsymbol{V} + \boldsymbol{A}\boldsymbol{r}$, where $\boldsymbol{V}(t)$ is a constant stream, $\boldsymbol{A}(t)$ is a second-rank tensor with zero trace and $\boldsymbol{r}(x_1, x_2, x_3)$ is a radius vector measured with respect to an arbitrary origin. It can be shown that

$$\boldsymbol{V}(t) + \boldsymbol{A}(t)\boldsymbol{r} = \boldsymbol{V}(t) + \tfrac{1}{2}\boldsymbol{\omega}_0(t) \times \boldsymbol{r} + \boldsymbol{E}(t)\boldsymbol{r}, \tag{2.1}$$

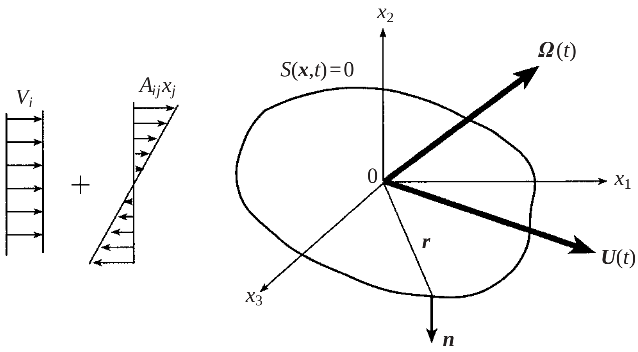

FIGURE 1.

where

$$E_{ij}(t) = \tfrac{1}{2}(A_{ij}(t) + A_{ji}(t)), \qquad \mathrm{tr}(\boldsymbol{E}(t)) = 0 \tag{2.2}$$

is a *symmetric* tensor and

$$\omega_{0i}(t) = -\varepsilon_{ijk}A_{jk}(t). \tag{2.3}$$

The indices i, j and k are either $1, 2$ or 3 and ε_{ijk} is the permutation tensor. The vorticity evolves with time according to the Helmholtz equation, i.e.

$$\frac{\mathrm{d}\boldsymbol{\omega}_0}{\mathrm{d}t} = \boldsymbol{E}\boldsymbol{\omega}_0, \tag{2.4}$$

which determines uniquely $\boldsymbol{\omega}_0(t)$ providing some initial conditions for the vorticity are prescribed (see for example Majda & Bertozzi 2002, § 1).

Let us next assume that a moving body is instantaneously introduced into the fluid such that the centre of mass of the homogeneous body coincides with the origin of coordinates. In general, the body moves with six degrees of freedom, i.e. translation $\boldsymbol{U}(t)$ of its centroid and angular velocity $\boldsymbol{\Omega}(t)$ of its principal axes (see Figure 1). Denoting the total induced velocity in the fluid by $\boldsymbol{v}$ and expressing the Euler equation in a moving (with the body) coordinate system, we obtain (see Kochin, Kibel & Rose 1965, p. 53)

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla\left(\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{U}_c\right) + \boldsymbol{\omega}\times(\boldsymbol{v} - \boldsymbol{U}_c) = -\nabla p, \tag{2.5}$$

where for simplicity we choose the fluid density as unity. Here, $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$ is the fluid vorticity and $\boldsymbol{U}_c$ denotes the total velocity of a point on the body surface S, i.e.

$$\boldsymbol{U}_c = \boldsymbol{U}(t) + \boldsymbol{\Omega}(t)\times\boldsymbol{r}. \tag{2.6}$$

In the following, we derive analytic expressions for both the hydrodynamic force $\boldsymbol{F}$ and moment $\boldsymbol{M}$ acting on the moving body in a vortical stream in terms of the dynamic pressure distribution p on the body surface, i.e.

$$\boldsymbol{F} = -\int_S p\boldsymbol{n}\,\mathrm{d}S, \qquad \boldsymbol{M} = -\int_S p\boldsymbol{r}\times\boldsymbol{n}\,\mathrm{d}S \tag{2.7}$$

where $\boldsymbol{n}$ is the outward normal to S directed into the unbounded fluid.

In two-dimensional flows, $\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t = 0$ and thus it is possible to replace $\boldsymbol{\omega}$ in (2.5) by the uniform ambient vorticity $\boldsymbol{\omega}_0$. On the other hand, for three-dimensional configurations, the choice of $\boldsymbol{\omega} = \boldsymbol{\omega}_0$ is valid only initially after the body is impulsively introduced into the fluid, since, because of three-dimensional vortex stretching effects,

$\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t \neq 0$. It can be shown, however, that the approximation of $\boldsymbol{\omega} \sim \boldsymbol{\omega}_0$ in three-dimensions, is still valid during a finite time interval satisfying $t^* < |\boldsymbol{\omega}_0| \,/\, |(\boldsymbol{\omega}_0 \cdot \nabla)\, \boldsymbol{v}|$, which can be also estimated as $t^* < (|\boldsymbol{\omega}_0| + (V_r/L))^{-1}$, where V_r is the module of the body's relative (to the fluid) velocity and L is its characteristic length scale. If the ambient vorticity field is weak with respect to the corresponding inertia scale, then we obtain $t^* < |\nabla^2_{2D} \Phi|^{-1}$. The linear operator ∇^2_{2D} represents the two-dimensional Laplacian in a plan orthogonal to the ambient vorticity vector. For genuine two-dimensional flows, this operator is identically null and thus $t^* < \infty$, implying that for planar flows the theory is valid for all time. The two-dimensional Laplacian can be considered also as an indication of three-dimensionality and the departure from the planar case.

The velocity vector $\boldsymbol{v}$ in (2.5) must satisfy the body impermeable boundary condition, i.e.

$$\boldsymbol{v} \cdot \boldsymbol{n}|_S = (\boldsymbol{U}_c + \boldsymbol{V}_d) \cdot \boldsymbol{n}|_S, \tag{2.8}$$

where $\boldsymbol{V}_d$ is the deformation velocity defined for a general deformable surface $S(\boldsymbol{r}, t) = 0$ as

$$\boldsymbol{V}_d \cdot \boldsymbol{n} = -\frac{\partial S/\partial t}{|\nabla S|}. \tag{2.9}$$

In order to satisfy the Neumann conditions of (2.8), let us express the fluid absolute velocity as

$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_\omega,$$

$$\boldsymbol{v}_0 = \boldsymbol{V} + \boldsymbol{E}\boldsymbol{r} + \nabla \Phi, \qquad \boldsymbol{v}_\omega = \tfrac{1}{2}\boldsymbol{\omega}_0 \times \boldsymbol{r} - \tfrac{1}{2}\omega_{0i} \nabla \varphi_{i+3} + \boldsymbol{v}_{vort}, \tag{2.10}$$

where $\boldsymbol{v}_{vort}$ is the response vortical velocity due to the presence of the body and Φ is a harmonic function defined in terms of the six-Kirchhoff potentials, φ_i and $\varphi_{i+3} (i = 1, 2, 3)$, as (e.g. Lamb 1932, chap. 6).

$$\Phi = (U_i - V_i)\varphi_i + \Omega_i \varphi_{i+3} + \phi_E + \phi_d, \tag{2.11}$$

where

$$\left.\frac{\partial \varphi_i}{\partial n}\right|_S = n_i, \quad \left.\frac{\partial \varphi_{i+3}}{\partial n}\right|_S = (\boldsymbol{r} \times \boldsymbol{n})_i, \quad \left.\frac{\partial \phi_E}{\partial n}\right|_S = -E_{ij} x_i n_j,$$
$$\boldsymbol{V}_d \equiv \nabla \phi_d, \qquad \boldsymbol{v}_{vort} \cdot \boldsymbol{n}|_S = 0. \tag{2.12}$$

It can then be easily verified that (2.10) and (2.11) indeed satisfy (2.8) by virtue of (2.12). If the ambient flow is irrotational, i.e. $\boldsymbol{\omega}_0 = 0$, then $\boldsymbol{v}_\omega = 0$ and the remaining velocity term $\boldsymbol{v}_0$ can be expressed in terms of a velocity potential.

In a similar manner, it is possible to split the total pressure into a potential p_0 and vortical p_ω parts, i.e. $p = p_0 + p_\omega$, where by substituting (2.10) into (2.5), we obtain

$$\frac{\partial \boldsymbol{v}_0}{\partial t} + \nabla\left(\tfrac{1}{2}\boldsymbol{v}_0 \cdot \boldsymbol{v}_0 - \boldsymbol{v}_0 \cdot \boldsymbol{U}_c\right) = -\nabla p_0 \tag{2.13}$$

and

$$\frac{\partial \boldsymbol{v}_\omega}{\partial t} + \nabla\left(\boldsymbol{v}_0 \cdot \boldsymbol{v}_\omega - \boldsymbol{U}_c \cdot \boldsymbol{v}_\omega + \tfrac{1}{2}\boldsymbol{v}_\omega \cdot \boldsymbol{v}_\omega\right) + \boldsymbol{\omega}_0 \times (\boldsymbol{v}_0 - \boldsymbol{U}_c + \boldsymbol{v}_\omega) = -\nabla p_\omega. \tag{2.14}$$

The vortical pressure term can again be decomposed into terms which are linear and quadratic in $\boldsymbol{\omega}_0$ by letting $p_\omega = p_\omega^{(1)} + p_\omega^{(2)}$ where,

$$\frac{\partial \boldsymbol{v}_\omega}{\partial t} + \nabla[\boldsymbol{v}_\omega \cdot (\boldsymbol{v}_0 - \boldsymbol{U}_c)] + \boldsymbol{\omega}_0 \times (\boldsymbol{v}_0 - \boldsymbol{U}_c) = -\nabla p_\omega^{(1)} \tag{2.15}$$

and

$$\nabla\left(\tfrac{1}{2}\boldsymbol{v}_\omega\cdot\boldsymbol{v}_\omega\right)+\boldsymbol{\omega}_0\times\boldsymbol{v}_\omega=-\nabla p_\omega^{(2)}. \tag{2.16}$$

Using these notations, the total hydrodynamic loads, (2.7), can be now written as,

$$\boldsymbol{F}=\boldsymbol{F}_0+\boldsymbol{F}_\omega^{(1)}+\boldsymbol{F}_\omega^{(2)},\qquad \boldsymbol{M}=\boldsymbol{M}_0+\boldsymbol{M}_\omega^{(1)}+\boldsymbol{M}_\omega^{(2)}. \tag{2.17}$$

The general (no restrictions imposed on the body's shape) solution for the irrotational contributions $\boldsymbol{F}_0$ and $\boldsymbol{M}_0$ for the ambient flow $\boldsymbol{V}+\boldsymbol{E}\boldsymbol{r}$, are well known (see Galper & Miloh 1994). For reasons of completeness, we reproduce and summarize the Galper & Miloh results, which are also valid for an arbitrary weakly non-uniform potential flow fields in Appendix A. In the next sections we determine the six remaining vortical load components ($\boldsymbol{F}_\omega$, $\boldsymbol{M}_\omega$) and their dependence on the body's six velocities ($\boldsymbol{U}$, $\boldsymbol{\Omega}$).

3. The vortical loads

In order to integrate the Euler equations, let us first rewrite (2.15), by using the notations of (2.10), as

$$\frac{\partial \boldsymbol{v}_\omega}{\partial t}+\nabla(\boldsymbol{v}_\omega\cdot\nabla\Phi)+\boldsymbol{\omega}_0\times\nabla\Phi-\tfrac{1}{2}\omega_{0j}\nabla(\nabla\varphi_{j+3}\cdot\boldsymbol{v}_r)+\nabla\left(\tfrac{1}{2}\boldsymbol{\omega}_0\times\boldsymbol{r}\cdot\boldsymbol{v}_r\right)+\boldsymbol{\omega}_0\times\boldsymbol{v}_r=-\nabla p_\omega^{(1)}, \tag{3.1}$$

where

$$\boldsymbol{v}_r=\boldsymbol{V}+\boldsymbol{E}\boldsymbol{r}-\boldsymbol{U}_c \tag{3.2}$$

represents the relative (to the body) velocity.

The acceleration term on the left-hand side of (3.1) is given by virtue of (2.10) as

$$\frac{\partial \boldsymbol{v}_\omega}{\partial t}=\frac{\partial \boldsymbol{v}_{vort}}{\partial t}+\tfrac{1}{2}\dot{\boldsymbol{\omega}}_0\times\boldsymbol{r}-\tfrac{1}{2}\dot{\omega}_{oj}\nabla\varphi_{j+3}-\tfrac{1}{2}\boldsymbol{\omega}_0\times(\boldsymbol{\omega}_0\times\boldsymbol{r})+\tfrac{1}{2}\boldsymbol{\omega}_0\times\omega_{oj}\nabla\varphi_{j+3}, \tag{3.3}$$

where the overdot denotes differentiation with respect to time. It has been demonstrated that during the initial stages ($t=0^+$), $\boldsymbol{v}_{vort}=0$, but generally for three-dimensional flows $\partial\boldsymbol{v}_{vort}/\partial t\neq 0$ since $\mathrm{d}\boldsymbol{\omega}_0/\mathrm{d}t\neq 0$. Moreover, by examining the various terms on the left-hand side of (3.1), we find that some of them (the first four terms) are singular within the body and some (the last two terms) are regular. Thus, it is possible again to split the linear pressure $p_\omega^{(1)}$ term into singular and regular parts, i.e.

$$p_\omega^{(1)}=p_{\omega S}^{(1)}+p_{\omega r}^{(1)}. \tag{3.4}$$

In order to find the corresponding pressure force we adopt a methodology similar to that originally proposed by Quartapelle & Napolitano (1982) (also see Howe 1995; Chang & Lei 1996; Wells 1998; Noca, Shields & Jeon 1999), by multiplying $\nabla p_{\omega S}$ by $\nabla\varphi_i$, integrate this product over the exterior (to the body) unbounded volume B_+ and subtract the integration of $\nabla p_{\omega r}$ over the interior body volume B_-. Using this procedure and by employing the Gauss theorem, we obtain the following expression for the force:

$$\int_{B_+}\nabla p_{\omega S}^{(1)}\cdot\nabla\varphi_i\,\mathrm{d}B-\int_{B_-}\nabla p_{\omega r}^{(1)}\,\mathrm{d}B=-\int_S\left(p_{\omega S}^{(1)}+p_{\omega r}^{(1)}\right)n_i\,\mathrm{d}S\equiv F_{\omega_i}^{(1)}. \tag{3.5}$$

The contributions from the regular pressure terms can be found easily from (3.1)–(3.3) by first verifying that

$$\int_{B_-}\left[\tfrac{1}{2}\dot{\boldsymbol{\omega}}_0\times\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{\Omega}\times(\boldsymbol{\omega}_0\times\boldsymbol{r})+\nabla\left(\tfrac{1}{2}\boldsymbol{\omega}_0\times\boldsymbol{r}\cdot\boldsymbol{v}_r\right)+\boldsymbol{\omega}_0\times\boldsymbol{v}_r\right]\mathrm{d}B=\tfrac{1}{2}B\boldsymbol{\omega}_0\times(\boldsymbol{V}-\boldsymbol{U}), \tag{3.6}$$

where B is the body's volume, since our choice of coordinate system implies that $\int_{B_-} x_i \,\mathrm{d}B = 0$.

Before dealing with the singular pressure terms, note first that the vortical acceleration term $\partial \boldsymbol{v}_{vort}/\partial t$, does not contribute to the force. In order to prove this claim, we substitute the first term in the right-hand side of (3.3) in the first integral in (3.5), recall that $\nabla \cdot \boldsymbol{v}_{vort} = 0$ and thus for a rigid body we obtain

$$-\int_{B_+} \frac{\partial \boldsymbol{v}_{vort}}{\partial t} \cdot \nabla \varphi_i \,\mathrm{d}B = -\int_{B_+} \nabla \cdot \left(\varphi_i \frac{\partial \boldsymbol{v}_{vort}}{\partial t} \right) \mathrm{d}B = \int_S \varphi_i \frac{\partial \boldsymbol{v}_{vort}}{\partial t} \cdot \boldsymbol{n} \,\mathrm{d}S = 0, \qquad (3.7)$$

since (see (2.12)) $\boldsymbol{v}_{vort} \cdot \boldsymbol{n}|_S = 0$ for all t and $\boldsymbol{v}_{vort}$ varies linearly with time for $t = 0^+$ (compare with Legendre & Magnaudet (1998) equation (A 4) in the case of a sphere).

The remaining singular terms in (3.1) can be next integrated (see Appendix B). Substituting these terms into (3.5) together with (3.6), we finally obtain the following expression for the vortical force,

$$\boldsymbol{F}_{\omega}^{(1)} = \tfrac{1}{2}\boldsymbol{Z}\dot{\boldsymbol{\omega}}_0 + \tfrac{1}{2}\boldsymbol{\Omega} \times \boldsymbol{Z}\boldsymbol{\omega}_0 + \tfrac{1}{2}B\boldsymbol{\omega}_0 \times (\boldsymbol{V} - \boldsymbol{U}) - \tfrac{1}{2}\boldsymbol{\omega}_0 \times \int_S \Phi \boldsymbol{n} \,\mathrm{d}S + \int_S (\boldsymbol{\omega}_0 \times \nabla \Phi \cdot \boldsymbol{n}) \boldsymbol{\varphi} \,\mathrm{d}S, \qquad (3.8)$$

where $\boldsymbol{Z}$ is the mixed added-mass tensor (see (A 1)) and $\boldsymbol{\varphi} \equiv (\varphi_1, \varphi_2, \varphi_3)$. Thus, once the velocity potential is known on S, i.e. $\Phi(S)$, we can easily determine the vortical force acting on an arbitrary shape directly from (3.8). Further simplifications for quadratic shapes are possible, as demonstrated in the following.

Using the same procedure, we can also readily obtain analytic expressions for the vortical moment exerted on the body. By using the decomposition of the hydrodynamic pressure into singular and regular parts, (3.4), it is possible to evaluate the following volume integrals which determine the vortical torque, i.e.

$$\int_{B_+} \nabla p_{\omega S}^{(1)} \cdot \nabla \varphi_{i+3} \,\mathrm{d}B + \int_{B_-} \nabla p_{\omega r}^{(1)} \times \boldsymbol{r} \,\mathrm{d}B = -\int_S \left(p_{\omega S}^{(1)} + p_{\omega r}^{(1)} \right) (\boldsymbol{r} \times \boldsymbol{n})_i \,\mathrm{d}S \equiv M_{\omega_i}^{(1)}. \qquad (3.9)$$

Let us first evaluate the contribution of the regular pressure terms in (3.9), where in an analogous manner to (3.6) and by using (3.2), we obtain

$$\int_{B_-} \nabla p_{\omega r}^{(1)} \times \boldsymbol{r} \,\mathrm{d}B = -\int_{B_-} \left[\tfrac{1}{2}\dot{\boldsymbol{\omega}}_0 \times \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{\Omega} \times (\boldsymbol{\omega}_0 \times \boldsymbol{r}) + \nabla \left(\tfrac{1}{2}\boldsymbol{\omega}_0 \times \boldsymbol{r} \cdot \boldsymbol{v}_r \right) + \boldsymbol{\omega}_0 \times \boldsymbol{v}_r \right] \times \boldsymbol{r} \,\mathrm{d}B. \qquad (3.10)$$

The various terms in the right-hand side of (3.10), can be computed easily and be expressed in terms of the general moment of inertia tensor of the body.

The next step is to calculate the first integral in the left-hand side of (3.9) which includes the singular pressure terms. The mathematical details can be found in Appendix C and here we represent only the final result for the vortical torque obtained from (3.9) and (3.10),

$$\boldsymbol{M}_{\omega}^{(1)} = \tfrac{1}{2}\boldsymbol{R}\dot{\boldsymbol{\omega}}_0 + \tfrac{1}{2}\boldsymbol{\Omega} \times \boldsymbol{R}\boldsymbol{\omega}_0 + \boldsymbol{K} - \tfrac{1}{2}\boldsymbol{\omega}_0 \times \int_S \Phi \boldsymbol{r} \times \boldsymbol{n} \,\mathrm{d}S + \int_S (\boldsymbol{\omega}_0 \times \nabla \Phi \cdot \boldsymbol{n}) \boldsymbol{\theta} \,\mathrm{d}S \qquad (3.11)$$

where $\boldsymbol{\theta} \equiv (\varphi_4, \varphi_5, \varphi_6)$ and $\boldsymbol{K}$ denotes the 'regular' torque given by the sum of (3.10) and (C 5), i.e.

$$\boldsymbol{K} = \tfrac{1}{2} \int_{B_-} \{ \boldsymbol{r} \times [(\dot{\boldsymbol{\omega}}_0 + \boldsymbol{\Omega} \times \boldsymbol{\omega}_0) \times \boldsymbol{r} + \boldsymbol{\omega}_0 \times \boldsymbol{E}\boldsymbol{r}] + (\boldsymbol{\omega}_0 \times \boldsymbol{r}) \times (\boldsymbol{\Omega} \times \boldsymbol{r} - \boldsymbol{E}\boldsymbol{r}) \} \,\mathrm{d}B.$$

Further simplifications arise in two-dimensional flows where the angular velocity $\boldsymbol{\Omega}$ and vorticity $\boldsymbol{\omega}_0$ vectors are colinear. These regular terms are represented by integrating the product $x_i x_j$ over the interior body volume B_- and thus can also be written in terms of the body's moment of inertia tensor. The similarity between (3.11) and (3.8) is worth mentioning. The tensor $\boldsymbol{R}$ denotes the rotational added-mass tensor, (A 1), and the first integral on the right-hand side of (3.11) can be also expressed in terms of the added-mass tensors $\boldsymbol{R}$ and $\boldsymbol{Z}$.

For completeness, we also provide here the corresponding quadratic vorticity load terms. It is important to note that these quadratic terms have been ignored in previous studies based on *a priori* assumption of 'weak' vorticity. It is emphasized again that the present formulation is free of any 'weak' assumptions and thus quadratic terms should be also included for the sake of consistency. These terms can be obtained in a similar way by splitting the left-hand side of (2.10) into singular and regular parts terms and performing the proper volume integration in (3.5) and (3.9) for $\nabla p_\omega^{(2)}$, leading to

$$\boldsymbol{F}_\omega^{(2)} = -\tfrac{1}{4}\boldsymbol{\omega}_0 \times \boldsymbol{Z}\boldsymbol{\omega}_0 - \tfrac{1}{2}\int_S (\boldsymbol{\omega}_0 \times \omega_{0j}\nabla\varphi_{j+3}\cdot\boldsymbol{n})\boldsymbol{\varphi}\,\mathrm{d}S \tag{3.12}$$

and

$$\boldsymbol{M}_\omega^{(2)} = -\tfrac{1}{4}\boldsymbol{\omega}_0 \times \boldsymbol{R}\boldsymbol{\omega}_0 - \tfrac{1}{4}\int_{B-}(\boldsymbol{\omega}_0\cdot\boldsymbol{r})(\boldsymbol{\omega}_0\times\boldsymbol{r})\,\mathrm{d}B - \tfrac{1}{2}\int_S (\boldsymbol{\omega}_0 \times \omega_{0j}\nabla\varphi_{j+3}\cdot\boldsymbol{n})\boldsymbol{\theta}\,\mathrm{d}S \tag{3.13}$$

since

$$[\boldsymbol{\omega}_0 \times (\boldsymbol{\omega}_0 \times \boldsymbol{r})] \times \boldsymbol{r} = (\boldsymbol{\omega}_0\cdot\boldsymbol{r})(\boldsymbol{\omega}_0 \times \boldsymbol{r}). \tag{3.14}$$

These quadratic in $\boldsymbol{\omega}_0$ terms clearly vanish for perfectly symmetric two-dimensional (cylinders) and three-dimensional (spheres) shapes (short-time) since both $\boldsymbol{\theta}$ and $\boldsymbol{r}\times\boldsymbol{n}$ are null on S. However, for non-symmetric shapes they render finite contributions unless the ambient vorticity is 'weak' and these quadratic terms $O(|\boldsymbol{\omega}_0|^2)$ are asymptotically small with respect to the linear terms.

Before concluding this section, let us consider in particular the case of steady two-dimensional flows with uniform (constant) ambient vorticity over rigid two-dimensional contours. Thus, we can use a stream function formulation even without imposing irrotationality. Under these conditions, the Euler equation, (2.5), renders a first integral (Bernoulli equation). We denote the two-dimensional (in the (x_1, x_2)-plane) velocity vector as

$$\boldsymbol{q} \equiv \boldsymbol{v} - \boldsymbol{U}_c = \left(\frac{\partial\Psi}{\partial x_2}, -\frac{\partial\Psi}{\partial x_1}\right), \tag{3.15}$$

where Ψ is the corresponding streamfunction (i.e. $\Psi|_s = 0$). For a constant vorticity field $\boldsymbol{\omega}_0$, the last term in (2.5) can be written simply as

$$\boldsymbol{\omega}_0 \times \boldsymbol{q} = \omega_0\nabla\Psi. \tag{3.16}$$

It follows then that the pressure everywhere in the fluid is given by

$$p = -\tfrac{1}{2}q^2 - \omega_0\Psi + \text{const.} \tag{3.17}$$

It appears that this expression was first derived by Proudman (1916). It is important to note, by virtue of the impermeability condition (2.8) applied on the rigid surface S, that $\boldsymbol{q}\cdot\boldsymbol{n}|_s = 0$. Thus, the only (tangential) component of $\boldsymbol{q}$ on S is $q_t = \partial\Psi/\partial n|_s$

where the streamfunction is given by

$$\Psi = -\tfrac{1}{2}\left(\Omega - \tfrac{1}{2}\omega_0\right)\left(x_1^2 + x_2^2 - 2\psi_4\right) + (\boldsymbol{U} - \boldsymbol{V})\boldsymbol{\cdot}(\psi_1 - x_2, \psi_2 + x_1) \\ - \tfrac{1}{2}E_{12}\left(x_1^2 - x_2^2\right) + \psi_E + \text{const}, \tag{3.18}$$

where (ψ_i, ψ_E) are the corresponding stream functions for the potentials (φ_i, ϕ_E). The constant in (3.18) is chosen so that $\Psi|_s = 0$. Thus, the total steady two-dimensional force and moment are simply given by

$$\boldsymbol{F} = \tfrac{1}{2}\int_S \left(\frac{\partial \Psi}{\partial n}\right)^2 \boldsymbol{n}\,\mathrm{d}S, \quad \boldsymbol{M} = \tfrac{1}{2}\int_S \left(\frac{\partial \Psi}{\partial n}\right)^2 (\boldsymbol{r}\times\boldsymbol{n})\mathrm{d}S, \tag{3.19a,b}$$

where the streamfunction in (3.18) appears to be that for an irrotational flow with $\boldsymbol{\Omega}$ replaced by $\boldsymbol{\Omega} - \frac{1}{2}\boldsymbol{\omega}_0$. This useful transformation was first reported by Miloh (1994). For two-dimensional flows, it can also be shown that the last integral terms in (3.8) and (3.11)–(3.13) can be expressed in terms of the corresponding streamfunction, i.e.

$$\int_S (\omega_0 \times \nabla\Phi \boldsymbol{\cdot} \boldsymbol{n})\boldsymbol{\varphi}\,\mathrm{d}S = \int_S \omega_0 \frac{\partial \Psi}{\partial n}\boldsymbol{\varphi}\,\mathrm{d}S = \int_S \omega_0 \Psi \boldsymbol{n}\,\mathrm{d}S, \tag{3.20}$$

since Ψ is harmonic. The connection between the left-hand side of (3.20) and the force contribution of the second term in the right-hand side of (3.17) for $\omega_0 = \text{const}$ is then obvious.

Summarizing, we have obtained general analytic expressions for calculating the hydrodynamic force and moment acting on a body of arbitrary shape moving unsteadily in a uniform ambient vortical flow field. The methodology is general in the sense that it provides a unified approach for treating both two-dimensional and three-dimensional cases. Some simplifications are possible for planar flows which result from using a streamfunction formulation. The total loads are decomposed into three terms in the manner described in (2.17). The 'irrotational' parts $(\boldsymbol{F}_0, \boldsymbol{M}_0)$ are given in Appendix A (see (A 6) and (A 9)) and the 'vortical' terms are given explicitly by (3.8) and (3.11)–(3.13). It is important to note that these expressions are general in the sense that no restrictions are imposed on the shape of the body, on its motion or on the magnitude of the ambient vorticity. These analytic expressions can be further reduced for quadratic shapes and written in terms of the various added-mass terms, as demonstrated in the next section.

4. Simplifications for quadratic symmetric shapes

The expressions for the vorticity induced force and moment acting on a general moving body can be further simplified for two-dimensional and three-dimensional quadratic shapes, which preserve a sign symmetry $S(X) = S(-X)$, such as circles, ellipses, spheres, spheroids and ellipsoids. These configurations often appear in numerous applications in fluid mechanics. A basic characteristic of quadratic shapes is that the added mass tensor, (A 1), is a purely diagonal second-order tensor and $\boldsymbol{Z} = \boldsymbol{Z}^T = 0$. Also, in both rectilinear $\boldsymbol{T}$ and angular $\boldsymbol{R}$ tensors, the off-diagonal terms are null. Another important property of such quadratic shapes is that the six Kirchhoff potentials, which are uniquely defined by solving a corresponding Neumann problem, can be expressed easily in terms of the added-mass tensor, i.e. (see for example Kochin *et al.* 1965 §7.8).

$$\varphi_i|_s = -\frac{1}{B}T_{ij}x_j \qquad (i = 1, 2, 3), \tag{4.1}$$

where $\boldsymbol{r}(x_1, x_2, x_3)$ denotes the radius vector of a point on S and B, as before, represents the volume of the body. In addition, we have

$$\varphi_{i+3}|_s = -Q_{ii} x_j x_k \qquad \text{(no sum)}, \tag{4.2}$$

where (i, j, k) are arranged in a cyclic order, such that $i \neq j \neq k \neq i$. The diagonal tensor $\boldsymbol{Q}$ is related to the angular added-mass $\boldsymbol{R}$ and the body's inertia $\boldsymbol{\chi}$ tensor by

$$Q_{ii} = \frac{R_{ii}}{\chi_{jj} - \chi_{kk}} \quad \text{(no sum)}, \qquad \chi_{jk} = \int_{B_-} x_j x_k \, \mathrm{d}B. \tag{4.3a,b}$$

Clearly, the inertia tensor χ_{jk} is diagonal and if, let us say, $\chi_{jj} = \chi_{kk}$ for some two integers $j \neq k$, then by definition the corresponding angular added-mass R_{ii} is zero, such as in the case of a three-dimensional axisymmetric body with respect to the x_i-axis. Recognizing the fact that φ_i depends linearly on x_j and φ_{i+3} depends quadratically on the product $x_j x_k$, when evaluated on s, enables us to perform the surface integration analytically and derive closed-form expressions for the vortical loads in terms of the added-mass tensor.

Let us first consider the case of a rigid body moving with six degrees of freedom in a uniform ambient vortical flow where the velocity field away from the body is given by $\boldsymbol{V} + \frac{1}{2}\boldsymbol{\omega}_0 \times \boldsymbol{r}$ (i.e. rigid-body rotation). No restrictions are imposed at this stage on the orientations of either $\boldsymbol{V}$ or $\boldsymbol{\omega}_0$. Since the ambient straining in this case is null (i.e. $\boldsymbol{E} = 0$) the total velocity potential can be written as (see (2.11)).

$$\Phi = (U_i - V_i)\varphi_i + \Omega_i \varphi_{i+3}. \tag{4.4}$$

Substituting (4.1)–(4.4) into the last integral on the right-hand side of (3.8) yields

$$\int_S (\boldsymbol{\omega}_0 \times \nabla\Phi \cdot \boldsymbol{n})\boldsymbol{\varphi} \, \mathrm{d}S = \frac{\boldsymbol{T}}{B} \left[(\boldsymbol{\omega}_0 \times \boldsymbol{T}(\boldsymbol{U} - \boldsymbol{V}) \right]. \tag{4.5}$$

Note that there is no contribution from the angular term $\Omega_i \varphi_{i+3}$ in (4.4) since, for quadratic shapes, the integral $\int_S x_i x_j n_k \, \mathrm{d}S$ is identically zero for any combination of (i, j, k).

Recalling that $\boldsymbol{Z} = 0$ for quadratic shapes and that the third term on the right-hand side of (3.8) can easily be expressed in terms of the rectilinear added-mass, we readily obtain for the vortical force,

$$\boldsymbol{F}_\omega = -\tfrac{1}{2} B \boldsymbol{\omega}_0 \times (\boldsymbol{U} - \boldsymbol{V}) + \left(\frac{\boldsymbol{I}}{2} + \frac{\boldsymbol{T}}{B} \right) [\boldsymbol{\omega}_0 \times \boldsymbol{T}(\boldsymbol{U} - \boldsymbol{V})], \tag{4.6}$$

where $\boldsymbol{I}$ denotes the identity matrix. It is also noteworthy that for such symmetric shapes the two surface integrals in (3.8) do not contain any contributions from ϕ_E and thus (4.6) is independent of the ambient strain $\boldsymbol{E}$. This special property is, however, valid only for quadratic shapes. Again, it appears that there are no interaction terms between the ambient vorticity $\boldsymbol{\omega}_0$ and the angular velocity of the body $\boldsymbol{\Omega}$ contributing to the force in (4.6). Moreover, it is rather remarkable to see that because of the quadratic form of φ_{i+3} (see (4.2)), the right-hand side of (3.12) vanishes owing to symmetry since $\boldsymbol{Z} = 0$ and $\int_{B_-} x_i x_j \, \mathrm{d}B = 0$ for $i \neq j$. Thus, $\boldsymbol{F}_\omega^{(2)} = 0$ and (4.6) in fact yields the total vortical force acting on a general quadratic shape in the case where the fluid far from the body is in rigid-body rotation ($\boldsymbol{E} = 0$).

Two trivial cases can now be easily examined; the first is a circular cylinder where the rectilinear added-mass coefficient is unity, i.e. $\boldsymbol{T} = S\delta_{ij}$ (a 2×2 matrix) and the cross-section area S replaces the body volume B. The vorticity vector $\boldsymbol{\omega}_0$ is pointing out of the plane. Using these values in (4.6) gives

$$F_\omega = S\boldsymbol{\omega}_0 \times (\boldsymbol{U} - \boldsymbol{V}). \tag{4.7}$$

Thus, the vortical force in this case is of a 'lift' nature with a lift coefficient $C_{L\omega} = 1$ (compare also with (1.2)), which is in full agreement with Batchelor (1967 equation (7.4.12)) as well as Auton *et al.* (1988) and Moreno-Inseretes, Ferriz-Mas & Schussker (1994).

The second elementary case is that of a sphere, where $\boldsymbol{T} = \frac{1}{2}B\delta_{ij}$ (a 3×3 matrix) and (4.6) gives

$$F_\omega = 0. \tag{4.8}$$

In other words, it is shown that unlike a cylinder, the lift coefficient for a sphere is identically zero ($C_{L\omega} = 0$). Thus, we are able to verify easily the numerical findings given in (A 21) of Legendre & Magnaudet (1998). When comparing the lift coefficients of a cylinder and a sphere, remember that the lift coefficient for a cylinder (as well as for any two-dimensional contour) is valid for all time, whereas that for a sphere is valid only during a small time interval before vortex stretching effects become dominant.

In order to demonstrate the versatility of the present solution, let us consider yet another example involving a two-dimensional elliptic cylinder with two major axes $a > b$ in rectilinear motion along its major axis. The corresponding two added masses for an ellipse are $T_{11} = \pi b^2$ and $T_{22} = \pi a^2$ (e.g. Lamb 1932, p. 85) and thus (4.6) yields

$$F_\omega = \tfrac{1}{2}S\left(1 + \frac{b}{a}\right)\boldsymbol{\omega}_0 \times (\boldsymbol{U} - \boldsymbol{V}),$$

where $S = \pi ab$, in agreement with the independent derivation based on pressure integration given in Appendix D using elliptic coordinates.

Similar expressions can also be obtained for the vortical torque acting on a quadratic body defined in (3.11), by making use of the special properties of the angular Kirchhoff potentials φ_{i+3} in (4.2). Thus, by omitting the tedious algebraic steps pertaining to the evaluation of the last integral in the right-hand side of (3.11), the latter equation can be simply written as

$$M_\omega^{(1)} = \tfrac{1}{2}\boldsymbol{R}\dot{\boldsymbol{\omega}}_0 + \tfrac{1}{2}\boldsymbol{\omega}_0 \times \boldsymbol{Q}\boldsymbol{\Omega} - \boldsymbol{Q}(\boldsymbol{\chi}\boldsymbol{\omega}_0 \times \boldsymbol{Q}\boldsymbol{\Omega}), \tag{4.9}$$

which implies that the steady torque is generated only by interaction effects between the ambient vorticity $\boldsymbol{\omega}_0$ and the angular velocity of the body $\boldsymbol{\Omega}$. The first-order torque is null for a non-rotating (translating) quadratic body. However, it can be shown that there exists a finite second-order torque, (3.13), even for a stationary body. Indeed, by virtue of the relationships (4.2) and (4.3), we can evaluate the various integrals in the right-hand side of (3.13), leading to

$$M_\omega^{(2)} = \tfrac{1}{4}\boldsymbol{\omega}_0 \times (\boldsymbol{\chi} - \boldsymbol{R})\boldsymbol{\omega}_0 + \tfrac{1}{2}\boldsymbol{Q}(\boldsymbol{\chi}\boldsymbol{\omega}_0 \times \boldsymbol{Q}\boldsymbol{\omega}_0). \tag{4.10}$$

It is clear that (4.9) and (4.10) vanish for both cylindrical and spherical shapes, since $\boldsymbol{\chi} = \boldsymbol{R} = 0$. For non-spherical shapes, the total vortical moment, in the case where the fluid at infinity is in rigid rotation with an angular velocity $\frac{1}{2}\boldsymbol{\omega}_0$, is given by the sum of (4.9) and (4.10).

After discussing the solution for the case of a 'rigid-body motion', let us next consider the important case of a simple shear motion, ($\boldsymbol{E} \neq 0$), where the ambient velocity at infinity is given by (2.1)–(2.3) with $\mathrm{tr}(\boldsymbol{A}) = 0$. Thus, for any prescribed linear velocity field, $\boldsymbol{\omega}_0$ and $\boldsymbol{E}$ are uniquely determined in terms of the second-order

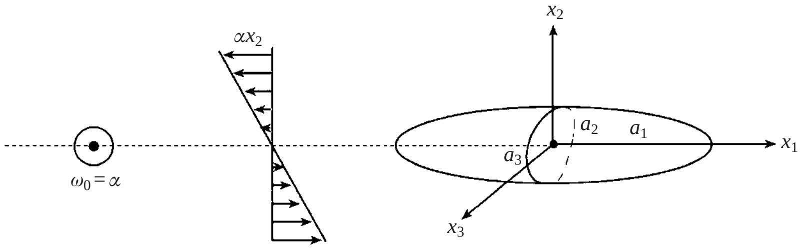

FIGURE 2.

tensor $\boldsymbol{A}$, where, in general, $\boldsymbol{E} = 0(\boldsymbol{\omega}_0)$. The $\boldsymbol{E}$-dependent irrotational force $\boldsymbol{F}_E$ and torque $\boldsymbol{M}_E$ exerted on a quadratic shape, can be found readily from (A 3), (A 6) and (A 9) as

$$\boldsymbol{F}_E = -(B\boldsymbol{I} + \boldsymbol{T})\boldsymbol{E}(\boldsymbol{U} - \boldsymbol{V}) + [\boldsymbol{T}, \boldsymbol{E}](\boldsymbol{U} - \boldsymbol{V}), \qquad \boldsymbol{M}_E = \boldsymbol{d} \times \boldsymbol{\Omega}, \tag{4.11}$$

where $\boldsymbol{d}$ is defined in (A 10) and (A 11) in terms of the strain tensor $\boldsymbol{E}$ and the angular Kirchhoff potential φ_{i+3}. The commutator operator $[\boldsymbol{T}, \boldsymbol{E}]$ is specified in (A 4) and it can be shown that it renders a force component perpendicular to $(\boldsymbol{U} - \boldsymbol{V})$, namely a lift force. A relation between this expression and the simplified non-uniform flow case presented in Saffman (1992 p. 88), can also be established.

In order to demonstrate this procedure for a uniform strain, let us study for example the case of a simple shear velocity with a single component along the x_1-axis (previously studied by Auton *et al.* (1988) and Legendre & Magnaudet (1998) for spherical shapes), given by (2.1), i.e.

$$(V - \alpha x_2)\boldsymbol{i} = V\boldsymbol{i} + \boldsymbol{E}\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{\omega}_0 \times \boldsymbol{r}, \tag{4.12}$$

where α is a parameter representing the dimensionless shear rate, and $\boldsymbol{i}$ is a unit vector in the x_1-direction. According to (2.3), $\boldsymbol{\omega}_0 = \alpha(0, 0, 1)$ and the only non-vanishing terms of the symmetric tensor $\boldsymbol{E}$ (defined in (2.2)), are $E_{12} = E_{21} = -\frac{1}{2}\alpha$. It can easily be seen then that the commutator operator $[\boldsymbol{T}, \boldsymbol{E}]$ vanishes for both cylindrical and spherical shapes and thus the so-called inertia (irrotational) force, (4.11), is a pure 'lift', i.e.

$$\begin{aligned}(\boldsymbol{F}_E)_{\text{cylinder}} = S\boldsymbol{\omega}_0 \times (\boldsymbol{U} - \boldsymbol{V}), (\boldsymbol{F}_E)_{\text{ellipse}} &= \tfrac{1}{2}S(1 + b/a)\boldsymbol{\omega}_0 \times (\boldsymbol{U} - \boldsymbol{V}),\\ (\boldsymbol{F}_E)_{\text{sphere}} &= \tfrac{3}{4}B\boldsymbol{\omega}_0 \times (\boldsymbol{U} - \boldsymbol{V}).\end{aligned} \tag{4.13}$$

It is also clear that $\boldsymbol{M}_E = 0$ for both shapes owing to symmetry, since $\varphi_{i+3} = 0$.

The total force acting on these perfectly symmetric shapes in a simple shearing motion is given now by adding (4.13) to (4.6) or (4.7). Thus, we one readily find that the 'shear' lift coefficient for a cylinder is $C_L = 2$ (see Batchelor 1967, p. 542, equation (7.4.15)) and the corresponding value for a sphere is $C_L = \frac{3}{4}$ (see Legendre & Magnaudet 1998, equation (A 15)), in agreement with (4.13).

In order to obtain a more general expression for the shear lift coefficient for non-spherical shapes, let us consider the case of a tri-axial ellipsoid,

$$\sum_{i=1}^{3} \left(\frac{x_i}{a_i}\right)^2 = 1 \tag{4.14}$$

moving with six degrees of freedom $(\boldsymbol{U}, \boldsymbol{\Omega})$ in an ambient shear flow defined in (4.12), where, for simplicity, we choose $\boldsymbol{V} = 0$ (see figure 2). Our aim is to evaluate analytically the vorticity induced force and moment components acting on the ellipsoid in terms of its six added-masses, i.e. (T_{11}, T_{22}, T_{33}) and (R_{11}, R_{22}, R_{33}). Substituting again $\boldsymbol{\omega}_0 = \alpha(0, 0, 1)$ and $E_{12} = E_{21} = -\frac{1}{2}\alpha$ in (4.11) the latter becomes

$$\boldsymbol{F}_E = \tfrac{1}{2}B(\boldsymbol{\omega}_0 \times \boldsymbol{U}) + \tfrac{1}{2}(\boldsymbol{\omega}_0 \times \boldsymbol{T}\boldsymbol{U}), \tag{4.15}$$

which simply reduces to (4.13) if $\boldsymbol{T} = S\delta_{ij}$ (cylinder) or $\boldsymbol{T} = \frac{1}{2}B\delta_{ij}$ (sphere).

Combining (4.15) with (4.6) and noting that also here $\boldsymbol{F}_\omega^{(2)} = 0$, owing to symmetry, yields the total shear force (keeping only terms proportional to $\boldsymbol{\omega}_0$),

$$\boldsymbol{F} = \left(\boldsymbol{I} + \frac{\boldsymbol{T}}{B}\right)(\boldsymbol{\omega}_0 \times \boldsymbol{T}\boldsymbol{U}). \tag{4.16}$$

It implies, for example, that if $\boldsymbol{U} = U(1, 0, 0)$, then the force is again a pure lift with a lift coefficient given by

$$C_L = \frac{T_{11}}{B}\left(1 + \frac{T_{22}}{B}\right), \tag{4.17}$$

which readily renders $C_L = 2$ (cylinder), $C_L = 1 + b/a$ (ellipse), and $C_L = \frac{3}{4}$ (sphere). It is important here to emphasize again the tensorial nature of the lift coefficient for non-perfectly symmetric shapes as compared with its scalar form for circular cylinders and spheres presented in Auton *et al.* (1988).

We turn next to computing the 'irrotational' moment given by (4.11). For this purpose we must evaluate the proportionality vector $\boldsymbol{d}$ by using (A 10). Recall again that for quadratic shapes the only non-zero terms which appear in the third-order tensor j_{ijk} defined in (A 11), are those for which $i \neq j \neq k \neq i$ (see Appendix A). Thus, since in the present case $E_{12} = E_{21} = -\frac{1}{2}\alpha$ and $\boldsymbol{\omega}_0 = \alpha(0, 0, 1)$, we find that $\boldsymbol{d}$ is collinear with $\boldsymbol{\omega}_0$ and thus

$$\boldsymbol{M}_E = \tfrac{1}{2}D\boldsymbol{\omega}_0 \times \boldsymbol{\Omega}, \qquad D \equiv Q_{11}(\chi_{22} + \chi_{33}) + Q_{22}(\chi_{33} + \chi_{11}) + Q_{33}(\chi_{11} + \chi_{22}), \tag{4.18}$$

which should be combined with (4.10) so as to render the expression for the total shear moment acting on the body. Here, $\boldsymbol{\chi}$ denotes the diagonal moment of inertia tensor defined in (4.3*b*) and Q is related to the diagonal rotational added-mass tensor $\boldsymbol{R}$ by (4.3*a*). The total moment experienced by a quadratic shape moving in the particular ambient stream, (4.12), is given by the sum of (4.9) and (4.18), since it is easily verified that in this case $\boldsymbol{M}_\omega^{(2)} = 0$, i.e.

$$\boldsymbol{M} = \tfrac{1}{2}\boldsymbol{R}\dot{\boldsymbol{\omega}}_0 + \tfrac{1}{2}\boldsymbol{\omega}_0 \times (D\boldsymbol{I} + \boldsymbol{Q})\boldsymbol{\Omega} - \boldsymbol{Q}(\boldsymbol{\chi}\boldsymbol{\Omega}_0 \times \boldsymbol{Q}\boldsymbol{\Omega}). \tag{4.19}$$

Finally, it is worth noting that if $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}_0$ are parallel, the last two terms on the right-hand side of (4.19) vanish and if, in addition, $\boldsymbol{\omega}_0$ does not depend on time, then the total moment $\boldsymbol{M}$ for a quadratic body is null. More general cases, involving for example non-symmetric bodies, where $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}_0$ are non-parallel or where $\boldsymbol{U}$ and $\boldsymbol{V}$ are non-collinear, can also be treated by the same methodology.

5. Summary and conclusions

General analytic expressions are derived for the pressure force and moment acting on a moving body of arbitrary shape in a linear shear flow with ambient uniform vorticity. The analysis is uniformly valid both for two-dimensional as well as for three-dimensional deformable shapes, the only exception being that in three-dimensions, the solution only holds initially (small time) owing to vortex stretching. First, we consider the case where the flow at infinity is in rigid-body motion (applications in turbo-machinery) with an angular velocity $\frac{1}{2}\boldsymbol{\omega}_0(t)$. The induced vorticity in the incompressible fluid is uniform and given by $\boldsymbol{\omega}_0(t)$. The vortical force and moment are expressed in terms of the ambient vorticity, the body's six velocities and its added-mass tensor. The vortical force is given by (3.8) + (3.12) and is represented by surface integrals over S involving the various Kirchhoff unit potentials. A similar expression is derived for the vortical torque, i.e. (3.11) + (3.13). It is important to note that there is no need to determine explicitly the induced vortical velocity or impose any weak conditions on the magnitude of $\boldsymbol{\omega}_0$. By making use of the special properties of the Kirchhoff potentials ((4.1) − (4.2)), which hold for quadratic (symmetric) shapes, it is possible to obtain simple analytic relationships for the force, (4.6), in terms of the rectilinear added-mass tensor $\boldsymbol{T}$, which contain only linear terms in $\boldsymbol{\omega}_0$. Since for such shapes $\boldsymbol{T}$ is purely diagonal, the force is of a 'lift' nature acting in a direction perpendicular to both the relative velocity $\boldsymbol{U} - \boldsymbol{V}$ and the ambient vorticity $\boldsymbol{\omega}_0$. It readily yields the well-known solutions for a cylinder and a sphere where $\boldsymbol{T}$ can be replaced by the two-dimensional or three-dimensional identity matrices, for which the lift-coefficient is a scalar. However, it is demonstrated that for non-perfectly symmetric shapes the lift coefficient is of a tensorial nature. Similar expressions can be found for the moment in terms of the rotational added-mass tensor $\boldsymbol{R}$ and the body's moment of inertia tensor $\boldsymbol{\chi}$, (4.3b). It includes both linear terms in $\boldsymbol{\omega}_0$, (4.9), and quadratic terms in $\boldsymbol{\omega}_0$, (4.10). If $\boldsymbol{\omega}_0$ and $\boldsymbol{\Omega}$ are directed along one of the major axes of the body, then $\boldsymbol{M}_\omega^{(1)}$ is perpendicular to both $\boldsymbol{\omega}_0$ and $\boldsymbol{\Omega}$ and $\boldsymbol{M}_\omega^{(2)} = 0$.

For a simple shearing motion with a linear velocity profile $\boldsymbol{A}\boldsymbol{r}$, (2.1), the prescribed tensor $\boldsymbol{A}$ determines uniquely both the ambient vorticity $\boldsymbol{\omega}_0$ and rate of strain tensor $\boldsymbol{E}$ ((2.2) and (2.3)). Thus, an 'irrotational' force component $\boldsymbol{F}_E$ has to be added to the previously found vortical force $\boldsymbol{F}_\omega$. The $\boldsymbol{E}$-dependent force $\boldsymbol{F}_E$ and moment $\boldsymbol{M}_E$, are readily obtained from the general Galper & Miloh expressions for the force, (A 6), and moment, (A 9). By definition, they consist of terms which are linear in the ambient vorticity $\boldsymbol{\omega}_0$ and for quadratic shapes they are simply given by (4.11).

Next, we considerd in some detail the case of uni-directional shear velocity, (4.12), and obtained the corresponding expressions for the total vortical force, (4.16), and moment, (4.18), exerted on the body. Note that if $\boldsymbol{T}$ is not a unity matrix and if $\boldsymbol{U}$ has at least two components along the major axes, then $\boldsymbol{F}$ is not a genuine lift force. The newly simplified expression for the lift coefficient, (4.17), found in this case, readily yields the corresponding known values for a cylinder ($C_L = 2$) and a sphere ($C_L = \frac{3}{4}$) and also shows the connection between the two. Even for such a simple uni-directional shear flow, the lift coefficient depends on products of the longitudinal and transverse added-masses. The unified methodology presented here also enables us to consider in a straightforward manner more complicated cases of body–fluid interactions.

Some claims made in several previous works (e.g. Rife *et al.* 1997; Magnaudet & Eames 2000, § 3), suggest that C_L (in the large time limit) is equal to the longitudinal added-mass even for non-spherical axisymmetric shapes. This assertion remains to be proved. Of course, the present study does not allow us to draw any conclusion about the validity of this conjecture for three-dimensional flows; however, such a statement is false in two-dimensions.

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Appendix A. The irrotational term F_0 and M_0

We present here the relevant main results of Galper & Miloh (1994 (hereinafter referred to as GM), 1995) for the pressure-induced force and moment components acting on a general body moving unsteadily with six degrees of freedom in a linear (weakly non-uniform) velocity field of an irrotational flow. The ambient stream is taken to be of the form $\boldsymbol{V}+\boldsymbol{E}\boldsymbol{r}$, where $\boldsymbol{E}$ represents a symmetric second-order tensor with zero trace (cf. (2.1)), such that the ambient velocity is solenoid and curl free. It is convenient to express the hydrodynamic loads exerted on the body in terms of the common 6×6 symmetric added-mass tensor $\boldsymbol{B}_{\alpha\beta}$ defined as,

$$B_{\alpha\beta}=B_{\beta\alpha}\equiv\begin{vmatrix}\boldsymbol{T} & \boldsymbol{Z}\\ \boldsymbol{Z}^T & \boldsymbol{R}\end{vmatrix}=-\int_S \varphi_\alpha\frac{\partial\varphi_\beta}{\partial n}\,\mathrm{d}S \qquad (\alpha,\beta=1,2,\dots,6), \tag{A 1}$$

where φ_α are the six Kirchhoff unit potentials (see (2.11)).

Let us first consider, for reasons of simplicity, rigid non-rotating ($\boldsymbol{\Omega}=0$) symmetric (under sign inversion) quadratic shapes for which $\boldsymbol{Z}=\boldsymbol{Z}^T=0$. The force is then simply given by (GM, (3.34)),

$$\boldsymbol{F}_0=(B\boldsymbol{I}+\boldsymbol{T})\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}-\boldsymbol{T}\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t}+[\boldsymbol{T},\boldsymbol{E}](\boldsymbol{U}-\boldsymbol{V}), \tag{A 2}$$

which can be also compared with Saffman (1992, p. 89). Here, $\boldsymbol{I}$ is the unitary (Kronecker) second-rank matrix. The Lagrangian time derivative is given by

$$\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}\equiv\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}+\boldsymbol{E}(\boldsymbol{V}-\boldsymbol{U}) \tag{A 3}$$

and

$$[\boldsymbol{T},\boldsymbol{E}]\equiv\boldsymbol{T}\boldsymbol{E}-\boldsymbol{E}\boldsymbol{T} \tag{A 4}$$

represents the commutator operator between two-second (symmetric in this case) tensors. The equivalent index form of (A 4) when acting on a vector $\boldsymbol{V}$, is

$$[\boldsymbol{T},\boldsymbol{E}]\,\boldsymbol{V}\equiv T_{ij}E_{jk}V_k-E_{ij}T_{jk}V_k, \tag{A 5}$$

which acts along a direction perpendicular to $\boldsymbol{V}$ (i.e. 'lift' force). It is important to note that (A 5) vanishes for any purely diagonal matrix where $T_{ij}=c\delta_{ij}$. Thus, for a spherical shape, where $c=\frac{1}{2}B$, the last term in (A 2) is null and equation (1.7) of Auton *et al.* (1988) is thus recovered. Clearly, for non-spherical shapes, the angular body velocity $\boldsymbol{\Omega}$ should also be taken into consideration and the force expression (A 2) for this case should be augmented according to GM as follows:

$$\begin{aligned}\boldsymbol{F}_0=(B\boldsymbol{I}+\boldsymbol{T})\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}-\boldsymbol{T}\left(\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t}+\boldsymbol{\Omega}\times\boldsymbol{U}\right)-\boldsymbol{\Omega}\times\boldsymbol{T}(\boldsymbol{U}-\boldsymbol{V})\\ +\boldsymbol{T}\boldsymbol{\Omega}\times(\boldsymbol{U}-\boldsymbol{V})+[\boldsymbol{T},\boldsymbol{E}](\boldsymbol{U}-\boldsymbol{V}).\end{aligned} \tag{A 6}$$

For such 'quadratic' symmetric shapes, the force does not contain any interaction terms between the angular velocity $\boldsymbol{\Omega}$ and the ambient strain tensor $\boldsymbol{E}$.

Finally, we also give below the additional term that appears in the force equation for arbitrary (non-symmetric) shapes, where the off-diagonal (3×3) tensors $\boldsymbol{Z}$ and $\boldsymbol{Z}^T$ are generally non-zero. It is convenient in this case to define first a new third-order tensor depending on the rectilinear Kirchhoff potentials φ_i and the body geometry, i.e.

$$s_{ijk}\equiv\int_S\varphi_i(x_jn_k+x_kn_j)\,\mathrm{d}S=s_{ikj}. \tag{A 7}$$

Thus, (A 6) must be augmented by the following term expressed in index form (see GM)

$$(F_i)_{add} = -Z_{ij}\frac{\mathrm{d}\Omega_j}{\mathrm{d}t} - \varepsilon_{ijk}\Omega_j Z_{kl}\Omega_l - E_{ij}Z_{jk}\Omega_k - \tfrac{1}{2}\varepsilon_{ijk}\Omega_j s_{klm}E_{lm}$$
$$- \tfrac{1}{2}s_{ijk}\frac{\partial E_{jk}}{\partial t} + \tfrac{1}{2}s_{ijk}(\varepsilon_{lkm}E_{jl} - \varepsilon_{jlm}E_{lk})\Omega_m, \qquad \text{(A 8)}$$

where ε_{ijk} is the common permutation tensor (i.e. $(\boldsymbol{A}\times\boldsymbol{B})_i = \varepsilon_{ijk}A_jB_k$).

These additional terms represent the more complicated mode of interaction between the ambient strain tensor $\boldsymbol{E}$ and the angular velocity $\boldsymbol{\Omega}$ which generally arise for a non-symmetric body. The most general unsteady case is given by (3.24)–(3.27) of GM.

What remains now is to provide the corresponding expression for the hydrodynamic torque. For the sake of brevity we reproduce here only the expression for the torque acting on a general 'quadratic' (symmetric) three-dimensional shape. By referring to (5.3) of GM, we obtain,

$$M_0 = -\boldsymbol{R}\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} - \boldsymbol{\Omega}\times\boldsymbol{R\Omega} - (\boldsymbol{U}-\boldsymbol{V})\times\boldsymbol{T}(\boldsymbol{U}-\boldsymbol{V}) + \boldsymbol{d}\times\boldsymbol{\Omega}, \qquad \text{(A 9)}$$

where the vector $\boldsymbol{d}$ is defined as (see GM (4.37))

$$d_i \equiv j_{jki}E_{jk} - j_{jjk}E_{ki} + \tfrac{1}{2}j_{ijk}E_{jk}. \qquad \text{(A 10)}$$

The newly introduced tensor $\boldsymbol{j}$ can be also expressed in terms of the angular Kirchhoff potentials φ_{i+3}, in an analogous manner to $\boldsymbol{s}$ (A 7), as

$$j_{ijk} \equiv \int_S \varphi_{i+3}(x_jn_k + x_kn_j)\,\mathrm{d}S = j_{ikj}. \qquad \text{(A 11)}$$

For further generalizations of the expressions for the hydrodynamic moment in the case of a non-symmetric body (i.e. $\boldsymbol{Z}\neq 0$) as well as for the case of a deformable time-dependent shape, see GM § 4(c).

Appendix B. Evaluating the first integral in the left-hand side of (3.5)

Using (3.1)–(3.3) in (3.5), the contributions of the singular pressure terms to the total force are found from the following integral:

$$\int_{B_+}\nabla p^{(1)}_{\omega S}\cdot\nabla\varphi_i\,\mathrm{d}B = \int_{B_+}\left[\tfrac{1}{2}\dot\omega_{0j}\nabla\varphi_{j+3} - \tfrac{1}{2}\nabla(\boldsymbol{\omega}_0\times\boldsymbol{r}\cdot\nabla\Phi) - \boldsymbol{\omega}_0\times\nabla\Phi\right.$$
$$\left. + \tfrac{1}{2}\omega_{0j}\nabla(\nabla\varphi_{j+3}\cdot(\nabla\Phi + \boldsymbol{v}_r))\right]\cdot\nabla\varphi_i\,\mathrm{d}B$$
$$= \tfrac{1}{2}\,\dot\omega_{0j}Z_{ij} + \int_S(\boldsymbol{\omega}_0\times\nabla\Phi\cdot\boldsymbol{n})\varphi_i\,\mathrm{d}S$$
$$+ \tfrac{1}{2}\int_S[\boldsymbol{\omega}_0\times\boldsymbol{r}\cdot\nabla\Phi - \omega_{0j}\nabla\varphi_{j+3}\cdot(\nabla\Phi + \boldsymbol{v}_r)]n_i\,\mathrm{d}S. \qquad \text{(B 1)}$$

In deriving (B 1), we have employed the Gauss theorem relating volume and surface integrals, by realizing that the integral over the surface S_∞ of infinite radius bounding B_+ is null owing to the proper decay of the various terms far from the body. In addition, we have used in (B 1) the definition of the mixed added-mass tensor $\boldsymbol{Z}$ defined in (A 1).

The last integral in the right-hand side of (B 1) can be further reduced by using the following theorems involving any pair (ϕ, ψ) of harmonic functions (see Landweber

& Miloh 1980, equation (28));

$$\int_S (\nabla\phi\cdot\nabla\psi)\boldsymbol{n}\,\mathrm{d}S = \int_S \left(\nabla\phi\frac{\partial\psi}{\partial n} + \nabla\psi\frac{\partial\phi}{\partial n}\right)\mathrm{d}S, \tag{B 2}$$

and

$$\int_S (\boldsymbol{\omega}_0\times\boldsymbol{r}\cdot\nabla\Phi)\boldsymbol{n}\,\mathrm{d}S = \int_S (\boldsymbol{\omega}_0\times\boldsymbol{r}\cdot\boldsymbol{n})\nabla\Phi\,\mathrm{d}S - \boldsymbol{\omega}_0\times\int_S \Phi\boldsymbol{n}\,\mathrm{d}S. \tag{B 3}$$

Noting first that $\omega_{0j}\nabla\varphi_{j+3}\cdot\boldsymbol{n}\,|_S = \boldsymbol{\omega}_0\cdot\boldsymbol{r}\times\boldsymbol{n}$ and $(\nabla\Phi+\boldsymbol{v}_r)\cdot\boldsymbol{n}\,|_S = 0$, by virtue of (2.8)–(2.12), it is also recalled that $\int_S \boldsymbol{v}_r(\boldsymbol{\omega}_0\cdot\boldsymbol{r}\times\boldsymbol{n})\,\mathrm{d}S = 0$ as a direct consequence of the Gauss theorem where $\boldsymbol{v}_r$ is given in (3.2). Using (B 2) and (B 3), the last integral in (B 1) is simply reduced to $-\frac{1}{2}\boldsymbol{\omega}_0\times\int_S\Phi\boldsymbol{n}\,\mathrm{d}S$ and thus we obtain,

$$\int_{B_+}\nabla p_{\omega S}^{(1)}\cdot\nabla\varphi_i\,\mathrm{d}B = \tfrac{1}{2}\dot{\omega}_{0j}Z_{ij} + \int_S(\boldsymbol{\omega}_0\times\nabla\Phi\cdot\boldsymbol{n})\varphi_i\,\mathrm{d}S - \tfrac{1}{2}\left(\boldsymbol{\omega}_0\times\int_S\Phi\boldsymbol{n}\,\mathrm{d}S\right)_i \tag{B 4}$$

which together with (3.6) finally leads to (3.8) for the vortical force.

Appendix C. Evaluating the first integral in the left-hand side of (3.9)

Let us evaluate the following integral in a similar manner to (B 1), i.e.

$$\begin{aligned}\int_{B_+}\nabla p_{\omega S}^{(1)}\cdot\nabla\varphi_{i+3}\,\mathrm{d}B &= \tfrac{1}{2}\dot{\omega}_{0j}R_{ij} + \int_S(\boldsymbol{\omega}_0\times\nabla\Phi\cdot\boldsymbol{n})\varphi_{i+3}\,\mathrm{d}S\\ &\quad + \tfrac{1}{2}\int_S\left[\boldsymbol{\omega}_0\times\boldsymbol{r}\cdot\nabla\Phi - \omega_{0j}\nabla\varphi_{j+3}\cdot(\nabla\Phi+\boldsymbol{v}_r)\right](\boldsymbol{r}\times\boldsymbol{n})_i\,\mathrm{d}S.\end{aligned} \tag{C 1}$$

The second integral in the right-hand side (C 1) can be further simplified by using the following two identities which hold for any pair (ϕ,ψ) of harmonic functions (see Landweber & Miloh (1980), equation (38));

$$\int_S(\nabla\phi\cdot\nabla\psi)(\boldsymbol{r}\times\boldsymbol{n})\,\mathrm{d}S = \int_S\left[(\boldsymbol{r}\times\nabla\phi)\frac{\partial\psi}{\partial n} + (\boldsymbol{r}\times\nabla\psi)\frac{\partial\phi}{\partial n}\right]\mathrm{d}S \tag{C 2}$$

and

$$\int_S(\boldsymbol{\omega}_0\times\boldsymbol{r}\cdot\nabla\Phi)(\boldsymbol{r}\times\boldsymbol{n})\,\mathrm{d}S = \int_S(\boldsymbol{\omega}_0\times\boldsymbol{r}\cdot\boldsymbol{n})(\boldsymbol{r}\times\nabla\Phi)\,\mathrm{d}S - \boldsymbol{\omega}_0\times\int_S\Phi\boldsymbol{r}\times\boldsymbol{n}\,\mathrm{d}S. \tag{C 3}$$

Again, by noting that $(\nabla\Phi+\boldsymbol{v}_r)\cdot\boldsymbol{n}\,|_S = 0$, the substitution of (C 2)–(C 3) into (C 1) gives

$$\begin{aligned}\int_{B_+}\nabla p_{\omega S}^{(1)}\cdot\nabla\varphi_{i+3}\,\mathrm{d}B &= \tfrac{1}{2}\dot{\omega}_{0j}R_{ij} + \int_S(\boldsymbol{\omega}_0\times\nabla\Phi\cdot\boldsymbol{n})\varphi_{i+3}\,\mathrm{d}S\\ &\quad - \tfrac{1}{2}\left(\boldsymbol{\omega}_0\times\int_S\Phi\boldsymbol{r}\times\boldsymbol{n}\,\mathrm{d}S\right)_i - \tfrac{1}{2}\int_S(\boldsymbol{r}\times\boldsymbol{v}_r)_i(\boldsymbol{\omega}_0\cdot\boldsymbol{r}\times\boldsymbol{n})\,\mathrm{d}S.\end{aligned} \tag{C 4}$$

The last integral in the right-hand side of (C 4) is regular within the body and thus can also be written by using (3.2) and the Gauss theorem as,

$$-\tfrac{1}{2}\int_{B_-}(\boldsymbol{\omega}_0\times\boldsymbol{r}\cdot\nabla)(\boldsymbol{r}\times\boldsymbol{v}_r)_i\,\mathrm{d}B. \tag{C 5}$$

Appendix D. Solution for a two-dimensional elliptic cylinder

We present here an independent solution for the case of an ellipse with semi-axes a and b moving with velocity U in an ambient uniform current V and constant vorticity field ω_0. Both U and V are directed along the major axis x_1 and let us also assume for simplicity that $\boldsymbol{\Omega} = \boldsymbol{E} = 0$. The corresponding streamfunction is then given by (see (3.18) for notation)

$$\Psi(x_1, x_2) = -\tfrac{1}{2}\omega_0\left[\psi_4(x_1, x_2) - \tfrac{1}{2}\left(x_1^2 + x_2^2\right)\right] + (U-V)[\psi_1(x_1, x_2) - x_2] + \text{const.} \quad \text{(D 1)}$$

In order to determine the various streamfunctions, it is convenient to use orthogonal elliptic coordinates (ζ, η) defined by,

$$x_1 + \mathrm{i}x_2 = (a^2 - b^2)^{1/2}\cosh(\zeta + \mathrm{i}\eta). \quad \text{(D 2)}$$

Thus, the fundamental ellipse is given by $\zeta = \zeta_0 = \text{const}$, where $\zeta_0 = \tanh^{-1}(b/a)$. Following Lamb (1932, pp. 83–89), we can express (D 1) in elliptic coordinates as

$$\begin{aligned}\Psi(\zeta, \eta) = &-(a^2 - b^2)^{1/2}(U - V)\left[\sinh\zeta - \frac{b}{a-b}\mathrm{e}^{-\zeta}\right]\sin\eta \\ &- \tfrac{1}{8}\omega_0\left[(a^2 - b^2)(\cosh 2\zeta + \cos 2\eta) - (a+b)^2\mathrm{e}^{-2\zeta}\cos 2\eta\right]. \end{aligned} \quad \text{(D 3)}$$

It can be shown then that on the ellipse $\Psi(\zeta_0, \eta) = -ab\omega_0 = \text{const}$. The two metric coefficients of the conformal transformation $(x_1, x_2) \to (\zeta, \eta)$, evaluated at $\zeta = \zeta_0$, are

$$h_\zeta^2 = h_\eta^2 = a^2\sin^2\eta + b^2\cos^2\eta. \quad \text{(D 4)}$$

Equation (D 3) yields

$$\begin{aligned}\left.\frac{\partial\Psi}{\partial\zeta}\right|_{\zeta=\zeta_0} = &-(a^2 - b^2)^{1/2}(U - V)\left[\cosh\zeta_0 + \frac{b}{a-b}\mathrm{e}^{-\zeta_0}\right]\sin\eta \\ &- \tfrac{1}{4}\omega_0\left[(a^2 - b^2)\sinh 2\zeta_0 + (a+b)^2\mathrm{e}^{-2\zeta_0}\cos 2\eta)\right] \end{aligned} \quad \text{(D 5)}$$

and thus, the tangential velocity on the ellipse is given by

$$q_t = \frac{1}{h_\zeta}\left.\frac{\partial\Psi}{\partial\zeta}\right|_{\zeta=\zeta_0} = -\frac{1}{h_\zeta}\left\{(U-V)(a+b)\sin\eta + \tfrac{1}{2}\omega_0\left[ab + \tfrac{1}{2}(a^2-b^2)\cos 2\eta\right]\right\}. \quad \text{(D 6)}$$

The hydrodynamic force acting on the ellipse according to (3.19) is then

$$\boldsymbol{F} = \tfrac{1}{2}\int_0^{2\pi} q_t^2\left(\frac{1}{h_\zeta}\frac{\partial x_1}{\partial\zeta}, \frac{1}{h_\zeta}\frac{\partial x_2}{\partial\zeta}\right)h_\eta\,\mathrm{d}\eta. \quad \text{(D 7)}$$

Substituting (D 6), we find that the only surviving term is a 'lift' component

$$\begin{aligned}F_2 &= \tfrac{1}{2}a(a+b)(U - V)\omega_0\int_0^{2\pi}\frac{ab + \frac{1}{2}(a^2 - b^2)\cos 2\eta}{a^2\sin^2\eta + b^2\cos^2\eta}\sin^2\eta\,\mathrm{d}\eta \\ &= \tfrac{1}{2}\pi b(a+b)(U - V)\omega_0. \end{aligned} \quad \text{(D 8)}$$

The corresponding moment according to (3.19b) is null as expected, owing to symmetry.

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